SPRING 2025 MATH 590: QUIZ 7 SOLUTIONS

Name:

1. Define what it means for $u_1, u_2, \ldots, u_n$ in $\mathbb{R}^n$ to be an orthonormal basis. (2 points)

Solution. $u_1, u_2, \ldots, u_n$ is an orthonormal basis if it is a basis consisting of mutually orthogonal vectors of length one.

2. For columns vectors $v, w \in \mathbb{R}^2$, define $\langle v, w\rangle := v^t \cdot \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \cdot w$. This gives a new inner product on $\mathbb{R}^2$. Show that the vectors $v = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ are orthogonal with respect to this inner product. (4 points)

Solution. $\langle v, w\rangle = \begin{pmatrix} -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 0.$

3. Find an orthogonal basis for the subspace of $\mathbb{R}^4$ spanned by the three independent vectors $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$. (4 points)

Solution. One applies the Gram-Schmidt process. Take $w_1 = v_1$. Then

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} \cdot w_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}.$$

Continuing the process

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} \cdot w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} \cdot w_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} - 0 \cdot w_1 - \frac{1}{\frac{3}{2}} \cdot \begin{pmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \\ 0 \end{pmatrix}.$$

The vectors w_1, w_2, w_3 give the required orthogonal basis.